Put $e = \frac{1}{e'}$ and $v = \left(\frac{1-e_1^2v_1^2}{1-v_1^2}\right)^{\frac{1}{2}}$, then by substitution,

$$dH_x = \frac{a(1-e_1^2)dv_1}{(1-v_1^2)^{\frac{3}{2}}(1-e_1^2v_1^2)^{\frac{1}{2}}}. \tag{2}$$

Let $A = v_1\left(\frac{1-e_1^2v_1^2}{1-v_1^2}\right)^{\frac{1}{2}}$, then

$$dH_x = adA - a\left(\frac{1-e_1^2v_1^2}{1-v_1^2}\right)^{\frac{1}{2}}dv_1 + \frac{a(1-e_1^2)dv_1}{(1-v_1^2)^{\frac{1}{2}}(1-e_1^2v_1^2)^{\frac{1}{2}}};$$

$$\therefore \qquad H_x = aA - aE(e_1, v_1) + \int \frac{a(1-e_1^2)dv_1}{(1-v_1^2)^{\frac{1}{2}}(1-e_1^2v_1^2)^{\frac{1}{2}}}, \tag{3}$$

where $E(e_1, v_1)$ denotes an elliptic arc, semimajor axis unity, eccentricity $e_1$ and abscissa $v_1$.

Let $e_1 = \frac{1-\sqrt{(1-e_2^2)}}{1+\sqrt{(1-e_2^2)}}$ and $v_1 = \frac{2v_2}{1-e_1^2}\left(\frac{1-v_2^2}{1-e_2^2v_2^2}\right)^{\frac{1}{2}}$, and we get

$$\frac{a(1-e_1^2)dv_1}{(1-v_1^2)^{\frac{1}{2}}(1-e_1^2v_1^2)} = 2ae_1dv_1 + 2a\left(\frac{1-e_1^2v_1^2}{1-v_1^2}\right)^{\frac{1}{2}}dv_1$$

$$-2a(1+e_1)\left(\frac{1-e_2^2v_2^2}{1-v_2^2}\right)^{\frac{1}{2}}dv_2. \tag{4}$$

$$\therefore \qquad H_x = aA + 2ae_1v_1 + aE(e_1, v_1) - 2a(1+e_1)E(e_2, v_2). \tag{5}$$

See *Wright's Solutions of the Cambridge Problems*, vol. ii, pp. 148—9.

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## *A PROBLEM IN LEAST SQUARES.*

BY R. J. ADCOCK, MONMOUTH, ILL.

FIND the most probable position of the straight line determined by the measured coordinates, each measure being equally good or of equal weight, $(x_1, y_1)$, $(x_2, y_2)$, ... $(x_n, y_n)$ of $n$ points, that is find $a$ and $b$ in the equation

$$y = ax + b. \tag{1}$$

Since $y_1 - ax_1 - b$, $y_2 - ax_2 - b$, ... $y_n - ax_n - b$ are the distances, parallel to the axis $y$, from the $n$ points to the required line,

$$\left[(y_1-ax_1-b)^2 + (y_2-ax_2-b)^2 + \ldots + (y_n-ax_n-b)^2\right]\frac{1}{1-a^2} = S(d_1^2) = u, \tag{2}$$

is the sum of the squares of the normals from the $n$ points to the required line, which sum by the principle of least squares, ANALYST, p. 183, Vol. IV, must be a minimum. Therefore

$$\frac{du}{db}=\frac{2nb-2S(y_1)+2aS(x_1)}{1+a^2}=0, \text{ or } S(y_1)-aS(x_1)-nb=0, \quad (3)$$

where $S(y_1)=y_1+y_2+\ldots+y_n$. And

$$\frac{du}{da}=\frac{2aS(x_1^2)-2S(x_1y_1)+2bS(x_1)}{1+a^2}$$
$$-\frac{[S(y_1^2)+a^2S(x_1^2)+nb^2-2aS(x_1y_1)-2bS(y_1)+2abS(x_1)]2a}{(1+a^2)^2}=0,$$

or $$a^2S(x_1y_1)-a^2bS(x_1)+a[S(x_1^2)-S(y_1^2)]-nab^2+bS(x_1)+2abS(y_1)$$
$$-S(x_1y_1)=0, \quad (4)$$

where $S(x_1y_1)=x_1y_1+x_2y_2+\ldots+x_ny_n$.

Eliminating $b$ from (3) and (4) and solving for $a$,

$$a=\frac{S(y_1^2)-S(x_1^2)+[S(x_1)]^2-[S(y_1)]^2}{2[S(x_1y_1)-S(x_1)S(y_1)]}$$
$$\pm\sqrt{\left[1+\left(\frac{S(y_1^2)-S(x_1^2)+[S(x_1)]^2-[S(y_1)]^2}{2[S(x_1y_1)-S(x_1)S(y_1)]}\right)^2\right]}. \quad (5)$$

Equation (3) shows that the line passes through the centre of gravity of the $n$ points, and therefore by (2) it must be a principal axis of them. And by (5) there are two positions of the line, at right angles to each other, which make $S(d_1^2)$ a minimum. The first value of $a$ which gives the least minimum is the one which the problem requires.

$x_1=1$, $y_1=1$, $x_2=3$, $y_2=2$, $x_3=5$, $y_3=4$, gives

$$a=\frac{1+4+16-1-9-25+(1+3+5)^2-(1+2+4)^2}{2[(1+6+20)-(1+3+5)(1+2+4)]}\pm\sqrt{\left(1+\frac{1}{16}\right)}$$
$$=-\tfrac{1}{4}\pm\tfrac{1}{4}\sqrt{17}=.78078 \text{ or } 1.2808.$$

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NOTE ON THE QUANTITY $g$, PAGE 25, BY PROF. JOHNSON.—The value of this quantity is

$$g_n=\tfrac{1}{3}(2^n\pm1);$$

for, beginning with $g_0=0$, we have, from the given relation

$$g_{n-1}=2g_n\pm1,$$

$g_1=1$, $g_2=2-1$, $g_3=2^2-2+1$, and in general $g_n=2^{n-1}-2^{n-2}+2^{n-3}\ldots.$, whence

$$g_n=\frac{2^n\pm1}{2+1}.$$

In the table an error occurs in the value of $g_{19}$ which introduces a cumulative error in the subsequent values, and another error was made in forming $g_{28}$; the final value should be

$$g_{50}=375{,}299{,}968{,}947{,}541.$$